OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT **4727/01**

MATHEMATICS

Further Pure Mathematics 3

THURSDAY 25 JANUARY 2007 Morning

Time: 1 hour 30 minutes

Additional Materials: Answer Booklet (8 pages)
List of Formulae (MF1)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- **You are reminded of the need for clear presentation in your answers.**

This document consists of **4** printed pages.

1 **(i)** Show that the set of numbers $\{3, 5, 7\}$, under multiplication modulo 8, does not form a group. [2]

(ii) The set of numbers $\{3, 5, 7, a\}$, under multiplication modulo 8, forms a group. Write down the value of a. [1]

(iii) State, justifying your answer, whether or not the group in part **(ii)** is isomorphic to the multiplicative group $\{e, r, r^2, r^3\}$, where e is the identity and $r^4 = e$. [2]

2 Find the equation of the line of intersection of the planes with equations

$$\mathbf{r}.(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 4 \qquad \text{and} \qquad \mathbf{r}.(\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}) = 6,$$

giving your answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. [5]

3 **(i)** Solve the equation $z^2 - 6z + 36 = 0$, and give your answers in the form $r(\cos\theta \pm \mathrm{i}\sin\theta)$, where $r > 0$ and $0 \leqslant \theta \leqslant \pi$. [4]

(ii) Given that Z is either of the roots found in part **(i)**, deduce the exact value of Z^{-3}. [3]

4 The variables x and y are related by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 - y^2}{xy}. \qquad \text{(A)}$$

(i) Use the substitution $y = xz$, where z is a function of x, to obtain the differential equation

$$x\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1 - 2z^2}{z}.$$ [3]

(ii) Hence show by integration that the general solution of the differential equation (A) may be expressed in the form $x^2(x^2 - 2y^2) = k$, where k is a constant. [6]

5 A multiplicative group G of order 9 has distinct elements p and q, both of which have order 3. The group is commutative, the identity element is e, and it is given that $q \neq p^2$.

(i) Write down the elements of a proper subgroup of G

(a) which does not contain q, [1]

(b) which does not contain p. [1]

(ii) Find the order of each of the elements pq and pq^2, justifying your answers. [3]

(iii) State the possible order(s) of proper subgroups of G. [1]

(iv) Find two proper subgroups of G which are distinct from those in part **(i)**, simplifying the elements. [4]

6 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} + 3y = 2x + 1.$$

Find

(i) the complementary function, [1]

(ii) the general solution. [5]

In a particular case, it is given that $\frac{dy}{dx} = 0$ when $x = 0$.

(iii) Find the solution of the differential equation in this case. [3]

(iv) Write down the function to which y approximates when x is large and positive. [1]

7 The position vectors of the points A, B, C, D, G are given by

$$\mathbf{a} = 6\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad \mathbf{c} = \mathbf{i} + 5\mathbf{j} + 4\mathbf{k}, \quad \mathbf{d} = 3\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}, \quad \mathbf{g} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$$

respectively.

(i) The line through A and G meets the plane BCD at M. Write down the vector equation of the line through A and G and hence show that the position vector of M is $2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$. [6]

(ii) Find the value of the ratio $AG : AM$. [1]

(iii) Find the position vector of the point P on the line through C and G, such that $\overrightarrow{CP} = \frac{4}{3}\overrightarrow{CG}$. [2]

(iv) Verify that P lies in the plane ABD. [4]

8 **(i)** Use de Moivre's theorem to find an expression for $\tan 4\theta$ in terms of $\tan\theta$. [4]

(ii) Deduce that $\cot 4\theta = \dfrac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$. [1]

(iii) Hence show that one of the roots of the equation $x^2 - 6x + 1 = 0$ is $\cot^2\left(\frac{1}{8}\pi\right)$. [3]

(iv) Hence find the value of $\operatorname{cosec}^2\left(\frac{1}{8}\pi\right) + \operatorname{cosec}^2\left(\frac{3}{8}\pi\right)$, justifying your answer. [5]

1 **(i)** Attempt to show no closure $3\times3=1,\ 5\times5=1$ *OR* $7\times7=1$	M1 A1	For showing operation table or otherwise For a convincing reason
OR Attempt to show no identity Show $a\times e=a$ has no solution	M1 A1 **2**	For attempt to find identity *OR* for showing operation table For showing identity is not 3, not 5, and not 7 by reference to operation table or otherwise
(ii) $(a=)\ 1$	B1 **1**	For value of a stated
(iii) *EITHER*: $\{e, r, r^2, r^3\}$ is cyclic, **(ii)** group is not cyclic	B1*	For a pair of correct statements
OR: $\{e, r, r^2, r^3\}$ has 2 self-inverse elements, **(ii)** group has 4 self-inverse elements	B1*	For a pair of correct statements
OR: $\{e, r, r^2, r^3\}$ has 1 element of order 2 **(ii)** group has 3 elements of order 2	B1*	For a pair of correct statements
OR: $\{e, r, r^2, r^3\}$ has element(s) of order 4 **(ii)** group has no element of order 4	B1*	For a pair of correct statements
Not isomorphic	B1 (dep*) **2** [5]	For correct conclusion
2 *EITHER*: $[3, 1, -2]\times[1, 5, 4]$ $\Rightarrow \mathbf{b}=k[1, -1, 1]$ e.g. put x *OR* y *OR* $z=0$ and solve 2 equations in 2 unknowns Obtain $[0, 2, -1]$ *OR* $[2, 0, 1]$ *OR* $[1, 1, 0]$	M1 A1 M1 M1 A1	For attempt to find vector product of both normals For correct vector identified with **b** For giving a value to one variable For solving the equations in the other variables For a correct vector identified with **a**
OR: Solve $3x+y-2z=4,\ x+5y+4z=6$ e.g. $y+z=1$ *OR* $x-z=1$ *OR* $x+y=2$ Put x *OR* y *OR* $z=t$ $[x, y, z]=[t, 2-t, -1+t]$ *OR* $[2-t, t, 1-t]$ *OR* $[1+t, 1-t, t]$ Obtain $[0, 2, -1]$ *OR* $[2, 0, 1]$ *OR* $[1, 1, 0]$ Obtain $k[1, -1, 1]$	 M1 M1 M1 A1 A1 **5** [5]	 For eliminating one variable between 2 equations For solving in terms of a parameter For obtaining a parametric solution for x, y, z For a correct vector identified with **a** For correct vector identified with **b**
3 **(i)** $z=\dfrac{6\pm\sqrt{36-144}}{2}$ $z=3\pm3\sqrt{3}\,\mathrm{i}$ Obtain $(r=)\ 6$ Obtain $(\theta=)\ \frac{1}{3}\pi$	M1 A1 A1 A1 **4**	For using quadratic equation formula or completing the square For obtaining cartesian values **AEF** For correct modulus For correct argument
(ii) *EITHER*: 6^{-3} *OR* $\frac{1}{216}$ seen $z^{-3}=6^{-3}(\cos(-\pi)\pm\mathrm{i}\sin(-\pi))$ Obtain $-\frac{1}{216}$	B1√ M1 A1	f.t. from their r^{-3} For using de Moivre with $n=\pm3$ For correct value
OR: $z^3=6z^2-36z=6(6z-36)-36z$ 216 seen Obtain $-\frac{1}{216}$	M1 B1 A1 **3** [7]	For using equation to find z^3 Ignore any remaining z terms For correct value

4 **(i)** $(y = xz \Rightarrow)\ \frac{dy}{dx} = x\frac{dz}{dx} + z$ $x\frac{dz}{dx} + z = \frac{x^2(1-z^2)}{x^2 z} = \frac{1}{z} - z$ $x\frac{dz}{dx} = \frac{1}{z} - 2z = \frac{1-2z^2}{z}$	B1 M1 A1 **3**	For a correct statement For substituting into differential equation and attempting to simplify to a variables separable form For correct equation **AG**
(ii) $\int \frac{z}{1-2z^2}\,dz = \int \frac{1}{x}\,dx \Rightarrow -\frac{1}{4}\ln(1-2z^2) = \ln cx$ $1-2z^2 = (cx)^{-4}$ $\frac{x^2-2y^2}{x^2} = \frac{c^{-4}}{x^4}$ $x^2(x^2-2y^2) = k$	M1 M1* A1 A1√ M1 (dep*) A1 **6** **9**	For separating variables and writing integrals For integrating both sides to ln forms For correct result (*c* not required here) For exponentiating their ln equation including a constant (this may follow the next M1) For substituting $z = \frac{y}{x}$ For correct solution properly obtained, including dealing with any necessary change of constant to *k* as given **AG**
5 **(i)** **(a)** e, p, p^2 **(b)** e, q, q^2	B1 B1 **2**	For correct elements For correct elements **SR** If the answers to parts **(i)** and **(iv)** are reversed, full credit may be earned for both parts
(ii) $p^3 = q^3 = e \Rightarrow (pq)^3 = p^3q^3 = e$ $\Rightarrow$ order 3 $(pq^2)^3 = p^3q^6 = p^3(q^3)^2 = e \Rightarrow$ order 3	M1 A1 A1 **3**	For finding $(pq)^3$ or $(pq^2)^3$ For correct order For correct order **SR** For answer(s) only allow B1 for either or both
(iii) 3	B1 **1**	For correct order and no others
(iv) e, pq, p^2q^2 *OR* $e, pq, (pq)^2$ e, pq^2, p^2q *OR* $e, pq^2, (pq^2)^2$ *OR* $e, p^2q, (p^2q)^2$	B1 B1 B1 B1 **4** **10**	For stating *e* and either *pq* or p^2q^2 For all 3 elements and no more For stating *e* and either pq^2 or p^2q For all 3 elements and no more

6 **(i)** (CF $m=-3\Rightarrow$) Ae^{-3x}	B1 **1**	For correct CF
(ii) ($y=$) $px+q$	B1	For stating linear form for PI (may be implied)
$\Rightarrow p+3(px+q)=2x+1$	M1	For substituting PI into DE (needs y and $\frac{dy}{dx}$)
$\Rightarrow p=\frac{2}{3},\quad q=\frac{1}{9}$	A1 A1	For correct values
$\Rightarrow$ GS $y=Ae^{-3x}+\frac{2}{3}x+\frac{1}{9}$	A1√	For correct GS. f.t. from their CF + PI
		SR Integrating factor method may be used, but CF must be stated somewhere to earn the mark in **(i)**
I.F. $e^{3x}\quad\Rightarrow\frac{d}{dx}\left(ye^{3x}\right)=(2x+1)e^{3x}$	B1	For stating integrating factor
$\Rightarrow ye^{3x}=\frac{1}{3}e^{3x}(2x+1)-\int\frac{2}{3}e^{3x}dx$	M1	For attempt at integrating by parts the right way round
$\Rightarrow ye^{3x}=\frac{2}{3}xe^{3x}+\frac{1}{3}e^{3x}-\frac{2}{9}e^{3x}+A$	A2 *	For correct integration, including constant Award A1 for any 2 algebraic terms correct
$\Rightarrow$ GS $y=Ae^{-3x}+\frac{2}{3}x+\frac{1}{9}$	A1√ **5**	For correct GS. f.t. from their * with constant
(iii) *EITHER* $\frac{dy}{dx}=-3Ae^{-3x}+\frac{2}{3}$	M1	For differentiating their GS
$\Rightarrow -3A+\frac{2}{3}=0$	M1	For putting $\frac{dy}{dx}=0$ when $x=0$
$y=\frac{2}{9}e^{-3x}+\frac{2}{3}x+\frac{1}{9}$	A1	For correct solution
OR $\frac{dy}{dx}=0,\ x=0\ \Rightarrow 3y=1$	M1	For using original DE with $\frac{dy}{dx}=0$ and $x=0$ to find y
$\Rightarrow\frac{1}{3}=A+\frac{1}{9}$	M1	For using their GS with y and $x=0$ to find A
$y=\frac{2}{9}e^{-3x}+\frac{2}{3}x+\frac{1}{9}$	A1 **3**	For correct solution
(iv) $y=\frac{2}{3}x+\frac{1}{9}$	B1√ **1**	For correct function. f.t. from linear part of **(iii)**
	10	

7 (i) *EITHER*: (**AG** is $\mathbf{r}$ =) $[6, 4, 8]+tk[1, 0, 1]$ *or* $[3, 4, 5]+tk[1, 0, 1]$	B1	For a correct equation
Normal to *BCD* is	M1	For finding vector product of any two of $\pm[1, -4, -1], \pm[2, 1, 1], \pm[1, 5, 2]$
$\mathbf{n} = k[1, 1, -3]$	A1	For correct $\mathbf{n}$
Equation of *BCD* is $\mathbf{r}.[1, 1, -3] = -6$	A1	For correct equation (or in cartesian form)
Intersect at $(6+t)+4+(-3)(8+t) = -6$	M1	For substituting point on *AG* into plane
$t = -4$ ($t = -1$ using $[3, 4, 5]$) $\Rightarrow$ $\mathbf{OM} = [2, 4, 4]$	A1	For correct position vector of *M* **AG**
OR: (**AG** is $\mathbf{r}$ =) $[6, 4, 8]+tk[1, 0, 1]$ *or* $[3, 4, 5]+tk[1, 0, 1]$	B1	For a correct equation
$\mathbf{r} = \mathbf{u}+\lambda\mathbf{v}+\mu\mathbf{w}$, where $\mathbf{u} = [2, 1, 3]$ *or* $[1, 5, 4]$ *or* $[3, 6, 5]$; $\mathbf{v}, \mathbf{w} =$ two of $[1, -4, -1], [1, 5, 2], [2, 1, 1]$	M1 A1	For a correct parametric equation of *BCD*
e.g. $(x =)\ 6+t = 2+\lambda+\mu$ $(y =)\ 4 = 1-4\lambda+5\mu$ $(z =)\ 8+t = 3-\lambda+2\mu$	M1	For forming 3 equations in t, λ, μ from line and plane, and attempting to solve them
$t = -4$ *or* $\lambda = -\frac{1}{3}, \mu = \frac{1}{3}$	A1	For correct value of t *or* λ, μ
$\Rightarrow$ $\mathbf{OM} = [2, 4, 4]$	A1 **6**	For correct position vector of *M* **AG**
(ii) A, G, M have $t = 0, -3, -4$ *OR* $AG = 3\sqrt{2}, AM = 4\sqrt{2}$ *OR* $\mathbf{AG} = [-3, 0, -3], \mathbf{AM} = [-4, 0, -4]$ $\Rightarrow AG : AM = 3 : 4$	B1 **1**	For correct ratio **AEF**
(iii) $\mathbf{OP} = \mathbf{OC}+\frac{4}{3}\mathbf{CG}$	M1	For using given ratio to find position vector of *P*
$= \left[\frac{11}{3}, \frac{11}{3}, \frac{16}{3}\right]$	A1 **2**	For correct vector
(iv) *EITHER*: Normal to *ABD* is	M1	For finding vector product of any two of $\pm[4, 3, 5], \pm[1, 5, 2], \pm[3, -2, 3]$
$\mathbf{n} = k[19, 3, -17]$	A1	For correct $\mathbf{n}$
Equation of *ABD* is $\mathbf{r}.[19, 3, -17] = -10$	M1	For finding equation (or in cartesian form)
$19.\frac{11}{3}+3.\frac{11}{3}-17.\frac{16}{3} = -10$	A1	For verifying that *P* satisfies equation
OR: Equation of *ABD* is $\mathbf{r} = [6, 4, 8]+\lambda[4, 3, 5]+\mu[1, 5, 2]$ (etc.)	M1	For finding equation in parametric form
$\left[\frac{11}{3}, \frac{11}{3}, \frac{16}{3}\right] = [6, 4, 8]+\lambda[4, 3, 5]+\mu[1, 5, 2]$	M1	For substituting *P* and solving 2 equations for λ, μ
$\lambda = -\frac{2}{3}, \mu = \frac{1}{3}$	A1	For correct λ, μ
	A1	For verifying 3rd equation is satisfied
OR: $\mathbf{AP} = \left[-\frac{7}{3}, -\frac{1}{3}, -\frac{8}{3}\right]$	M1 A1	For finding 3 relevant vectors in plane *ABDP* For correct **AP** *or* **BP** *or* **DP**
$\mathbf{AB} = [-4, -3, -5]$, $\mathbf{AD} = [-3, 2, -3]$ $\Rightarrow \mathbf{AB}+\mathbf{AD} = [-7, -1, -8]$	M1	For finding **AB, AD** *or* **BA, BD** *or* **DB, DA**
$\Rightarrow \mathbf{AP} = \frac{1}{3}(\mathbf{AB}+\mathbf{AD})$	A1 **4**	For verifying linear relationship
	13	

8 (i) $\cos 4\theta + i\sin 4\theta =$ $c^4 + 4ic^3s - 6c^2s^2 - 4ics^3 + s^4$ $\Rightarrow \sin 4\theta = 4c^3s - 4cs^3$ and $\cos 4\theta = c^4 - 6c^2s^2 + s^4$ $\Rightarrow \tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$	M1 A1 M1 A1 **4**	For using de Moivre with $n = 4$ For both expressions For expressing $\dfrac{\sin 4\theta}{\cos 4\theta}$ in terms of c and s For simplifying to correct expression
(ii) $\cot 4\theta = \dfrac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$	B1 **1**	For inverting **(i)** and using $\cot\theta = \dfrac{1}{\tan\theta}$ or $\tan\theta = \dfrac{1}{\cot\theta}$. **AG**
(iii) $\cot 4\theta = 0$ Put $x = \cot^2\theta$ $\theta = \frac{1}{8}\pi \Rightarrow x^2 - 6x + 1 = 0$ *OR* $x^2 - 6x + 1 = 0 \Rightarrow \theta = \frac{1}{8}\pi$	B1 B1 B1 **3**	For putting $\cot 4\theta = 0$ (can be awarded in **(iv)** if not earned here) For putting $x = \cot^2\theta$ in the numerator of **(ii)** For deducing quadratic from **(ii)** and $\theta = \frac{1}{8}\pi$ *OR* For deducing $\theta = \frac{1}{8}\pi$ from **(ii)** and quadratic
(iv) $4\theta = \frac{3}{2}\pi$ *OR* $\frac{1}{2}(2n+1)\pi$ 2nd root is $x = \cot^2\left(\frac{3}{8}\pi\right)$ $\Rightarrow \cot^2\left(\frac{1}{8}\pi\right) + \cot^2\left(\frac{3}{8}\pi\right) = 6$ $\Rightarrow \operatorname{cosec}^2\left(\frac{1}{8}\pi\right) + \operatorname{cosec}^2\left(\frac{3}{8}\pi\right) = 8$	M1 A1 M1 M1 A1 **5** **13**	For attempting to find another value of θ For the other root of the quadratic For using sum of roots of quadratic For using $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$ For correct value